

A PROFIT-MAXIMIZATION MODEL FOR A COMPANY THAT SELLS AN ARBITRARY NUMBER OF PRODUCTS

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ABSTRACT. One of the problems faced by a firm that sells certain commodities is to determine the number of products that it must supply in order to maximize its profit. In this article, the authors give an answer to this problem of economic interest. The proposed problem is a generalization of the results obtained by Stirzaker (Probability and Random Variables: A Beginner's Guide, 1999) and Kupferman (Lecture Notes in Probability, 2009) where the authors do not present a situation where the sale of a quantity from some commodities is constrained by the marketing of another. In addition, the described procedure is simple and can be successfully applied to any number of commodities. The obtained results can be easily put into practice.

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1. INTRODUCTION

Maximization problems arise in various fields of science, either in a direct form or in an indirect one. The objective of this paper is to offer an answer to the problem of maximizing the profit of a company that sells certain products in a competitive market under the assumption that previous statistics conducted by the firm, establish, as accurately as possible, the probability of purchasing products. More precisely, we will prove the following results:

Theorem 1. *Let us suppose a company wants to supply with two commodities: M_i , $i = 1, 2$, whose sale on the market brings the company a profit of c_i Euros/product if the product sells and a loss of s_i Euros/product if the product does not sell. Moreover, let X be a continuous random variable with the density function $f_X(x)$ that summarises the demand for commodity M_1 and distribution function $F_X(x)$ and Y a continuous random variable with the density function $f_Y(y)$ that summarises the demand for commodity M_2 with the distribution function $F_Y(y)$ and (X, Y) a continuous random vector with the density function $f_{X,Y}(x, y)$. If n_i ($i = 1, 2$) is the number of products from the commodity M_i that are about to be ordered by the company then the company maximizes its profit when n_1 is the point where the distribution function $F_X(x)$ reaches the level $c_1/(c_1 + s_1)$ the way x grows and n_2 is the point where the distribution function $F_Y(y)$ reaches the level $c_2/(c_2 + s_2)$ the way y grows.*

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We note that another problem facing us is to choose the number of products, given the profit function and, given the "constraints" imposed by certain factors that may occur. We formulate, thus, the following problem:

Problem 1. *If there are restrictions for n_1 and n_2 such as $f(n_1, n_2) \leq 0$ let us write the maximization problem in order to determine the quantity (n_1, n_2) from the commodities (M_1, M_2) you have to order so that the company will maximize the profit and to suggest a method to solve the above formulated problem.*

Note that the inequality $f(n_1, n_2) \leq 0$ summarises constraints of the number of products due to the problems faced by the firm.

Since in Theorem 1 the application is represented if the random variables are continuous, it is natural to ask what happens if they are discrete.

The answer is given in the following theorem:

Theorem 2. *Let us suppose that a company wants to supply with two commodities: M_i , $i = 1, 2$, whose marketing brings a profit of c_i Euros/product if the product sells and a loss of s_i Euros/product if the product does not sell. Let X be a discrete random variable with probability mass function $p_X(x)$ that summarises the demand for commodity M_1 , Y , a discrete random variable with probability mass function $p_Y(y)$ that summarises the demand for commodity M_2 and (X, Y) a discrete random vector with probability common mass function $p(i, j) = P(X = i, Y = j)$, $i, j = 0, 1, \dots$*

If n_i ($i = 1, 2$) is the number of products from the commodity M_i that are about to be ordered by the company then:

i) If the company net profit resulted from the marketing of the two commodity M_1, M_2 is given by the real function $g_{X,Y}(n_1, n_2)$ then the following equality takes place

$$M[g_{X,Y}(n_1, n_2)] = M[g_1(n_1, X)] + M[g_2(n_2, Y)]$$

where $g_1(n_1, X)$ is the net profit brought to the company by the commodity M_1 and $g_2(n_2, Y)$ is the net profit brought to the company by the commodities M_2 .

ii) As long as

$$(c_1 + s_1)P(X \leq n_1) < c_1 \text{ and } (c_2 + s_2)P(Y \leq n_2) < c_2$$

the company will maximize the profit.

Let us notice that, in the situation in which there are restrictions imposed on n_1 and n_2 like $f(n_1, n_2) = 0$ then the maximum number of products (n_1, n_2) , from the commodities M_1 and M_2 that must be ordered in order for the company to maximize the profit, is determined by

$$\max_{n_1, n_2 \geq 0} M[g_{X,Y}(n_1, n_2)] \text{ with constraints } f(n_1, n_2) = 0. \quad (1.1)$$

Theorem 2 leads to the following problem:

Problem 2. *Can we give an answer at the maximization problem (1.1)?*

The questions one and two appear, for example, inside the companies that produce both systems for video games and video games. For instance, the games can be compatible with older systems, which means that even if the company does not sell new systems, it will still have demands for the new games. Thus, the constraints of production, supply and transport for the systems imply the fact

that for a large number of sold systems there is a light decrease in the amount of profit per each sold game and therefore, taking into consideration the company interest to maximize the profit, the relations like $f(n_1, n_2) \leq 0$ are a must.

Remark 1: *The optimization problem we consider in **Theorem 1** and **Theorem 2** is to maximize the expected profit from selling some commodities. Other extensions of this work encompasses optimization criterions which involve higher moments of profit such as variance, skewness, kurtosis. For instance an interesting problem is to minimize the variance of the profit. We leave these extensions for future research.*

Let us note that the results from Theorems 1 and 2 are also found in a particular way in Stirzaker [2] and Kupferman [3].

We are in the position to give an answer at the mentioned theorems.

2. PROOF OF THEOREM 1

i) The net profit brought to the company by the two commodities is given by

$$g_{X,Y}(n_1, n_2) = \begin{cases} Xc_1 - (n_1 - X)s_1 + Yc_2 - (n_2 - Y)s_2 & \text{if } X \leq n_1, Y \leq n_2 \\ Xc_1 - (n_1 - X)s_1 + c_2n_2 & \text{if } X \leq n_1 \text{ and } Y > n_2 \\ c_1n_1 + Yc_2 - (n_2 - Y)s_2 & \text{if } X > n_1 \text{ and } Y \leq n_2 \\ c_1n_1 + c_2n_2 & \text{if } X > n_1 \text{ and } Y > n_2. \end{cases}$$

Or, written in a different way

$$g_{X,Y}(n_1, n_2) = \begin{cases} X(c_1 + s_1) + Y(c_2 + s_2) - n_1s_1 - n_2s_2 & \text{if } X \leq n_1, Y \leq n_2 \\ X(c_1 + s_1) - n_1s_1 + c_2n_2 & \text{if } X \leq n_1 \text{ and } Y > n_2 \\ Y(c_2 + s_2) + c_1n_1 - n_2s_2 & \text{if } X > n_1 \text{ and } Y \leq n_2 \\ c_1n_1 + c_2n_2 & \text{if } X > n_1 \text{ and } Y > n_2. \end{cases} \quad (2.1)$$

The expected gain is deduced using the law of the unconscious statistician, by calculating

$$\begin{aligned} M[g_{X,Y}(n_1, n_2)] &= \int_{-\infty}^{n_1} \int_{-\infty}^{n_2} [x(c_1 + s_1) + y(c_2 + s_2) - n_1s_1 - n_2s_2] f_{X,Y}(x, y) dx dy \\ &\quad + \int_{-\infty}^{n_1} \int_{n_2}^{\infty} [x(c_1 + s_1) - n_1s_1 + c_2n_2] f_{X,Y}(x, y) dy dx \\ &\quad + \int_{n_1}^{\infty} \int_{-\infty}^{n_2} [y(c_2 + s_2) + c_1n_1 - n_2s_2] f_{X,Y}(x, y) dy dx \\ &\quad + \int_{n_1}^{\infty} \int_{n_2}^{\infty} (c_1n_1 + c_2n_2) f_{X,Y}(x, y) dx dy \end{aligned}$$

or, echivalently

$$\begin{aligned}
M[g_{X,Y}(n_1, n_2)] &= \int_{-\infty}^{n_1} [xc_1 - (n_1 - x)s_1] f_X(x) dx + \int_{n_1}^{\infty} n_1 c_1 f_X(x) dx \\
&\quad + \int_{-\infty}^{n_2} [yc_2 - (n_2 - y)s_2] f_Y(y) dy + \int_{n_2}^{\infty} n_2 c_2 f_Y(y) dy \\
&= c_1 n_1 + (c_1 + s_1) \int_0^{n_1} (x - n_1) f_X(x) dx \\
&\quad + c_2 n_2 + (c_2 + s_2) \int_0^{n_2} (y - n_2) f_Y(y) dy.
\end{aligned}$$

We need to maximize this expression with respect to n_1 and n_2 . The simplest way to do it is find the critical point for this expression $M[g_{X,Y}(n_1, n_2)]$, thus

$$\begin{cases} M'_{n_1}[g_{X,Y}(n_1, n_2)] = 0 \\ M'_{n_2}[g_{X,Y}(n_1, n_2)] = 0 \end{cases}$$

equivalently

$$\begin{cases} c_1 + (c_1 + s_1) \int_0^{n_1} f_X(x) dx = c_1 - (c_1 + s_1) F_X(n_1) = 0 \\ c_2 + (c_2 + s_2) \int_0^{n_2} f_Y(y) dy = c_2 - (c_2 + s_2) F_Y(n_2) = 0 \end{cases}$$

from which we get that the critical point verifies the following

$$F_X(n_1) = c_1 / (c_1 + s_1) \text{ and } F_Y(n_2) = c_2 / (c_2 + s_2). \quad (2.2)$$

In order to establish if the determined critical point is maximum, we write the hessian matrix

$$H_M(n_1, n_2) = \begin{pmatrix} -(c_1 + s_1) f_X(n_1) & 0 \\ 0 & -(c_2 + s_2) f_Y(n_2) \end{pmatrix}.$$

We observe that $\Delta_1 = -(c_1 + s_1) f_X(n_1) < 0$ and

$$\begin{aligned}
\Delta_2 &= \begin{vmatrix} -(c_1 + s_1) f_X(n_1) & 0 \\ 0 & -(c_2 + s_2) f_Y(n_2) \end{vmatrix} \\
&= (c_1 + s_1)(c_2 + s_2) f_X(n_1) f_Y(n_2) > 0.
\end{aligned}$$

From which we deduce that the function $M[g_{X,Y}(n_1, n_2)]$ is strictly concave, meaning that the critical point obtained from (2.2) is the global maximum point for which the company should order n_1 products from the commodities M_1 and n_2 products from the commodities M_2 , where n_1 is the point in which the distribution function $F_X(x)$ reaches the level $c_1 / (c_1 + s_1)$ the x way grows and n_2 is the point in which the distribution function $F_Y(y)$ reaches the level $c_2 / (c_2 + s_2)$ the y way grows.

Question 1 boils down to proving

$$\max_{n_1, n_2 \geq 0} M[g_{X,Y}(n_1, n_2)] \text{ with constraint } f(n_1, n_2) \leq 0.$$

That can be solved, for example, using Lagrange multipliers method if $f(n_1, n_2) = 0$ or in the situation $f(n_1, n_2) < 0$ using other methods that can be found in reference of Intriligator [1].

3. PROOF OF THEOREM 2

i) The net profit of the company is given by the function $g_{X,Y}(n_1, n_2)$ defined through

$$g_{X,Y}(n_1, n_2) = \begin{cases} X(c_1 + s_1) + Y(c_2 + s_2) - n_1 s_1 - n_2 s_2 & \text{if } X = \overline{0, n_1}, Y = \overline{0, n_2} \\ X(c_1 + s_1) - n_1 s_1 + c_2 n_2 & \text{if } X = 0, \dots, n_1 \text{ and } Y > n_2 + 1 \\ Y(c_2 + s_2) + c_1 n_1 - n_2 s_2 & \text{if } X > n_1 \text{ and } Y = 0, \dots, n_2 \\ c_1 n_1 + c_2 n_2 & \text{if } X > n_1 \text{ and } Y > n_2. \end{cases} \quad (3.1)$$

Using **The extended law of the unconscious statistician** we deduce that the realised profit mean value by the company through the two products marketing is

$$\begin{aligned} M[g_{X,Y}(n_1, n_2)] &= \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} [i(c_1 + s_1) + j(c_2 + s_2) - n_1 s_1 - n_2 s_2] p(i, j) \\ &\quad + \sum_{i=0}^{n_1} \sum_{j=n_2+1}^{\infty} [i(c_1 + s_1) - n_1 s_1 + c_2 n_2] p(i, j) \\ &\quad + \sum_{i=n_1+1}^{\infty} \sum_{j=0}^{n_2} [j(c_2 + s_2) + c_1 n_1 - n_2 s_2] p(i, j) \\ &\quad + \sum_{i=n_1+1}^{\infty} \sum_{j=n_2+1}^{\infty} (c_1 n_1 + c_2 n_2) p(i, j) \\ &= (c_1 + s_1) \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} i p(i, j) + (c_2 + s_2) \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} j p(i, j) \\ &\quad - (n_1 s_1 + n_2 s_2) \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} p(i, j) \\ &\quad + (c_1 + s_1) \sum_{i=0}^{n_1} \sum_{j=n_2+1}^{\infty} i p(i, j) + (-n_1 s_1 + c_2 n_2) \sum_{i=0}^{n_1} \sum_{j=n_2+1}^{\infty} p(i, j) \\ &\quad + (c_2 + s_2) \sum_{i=n_1+1}^{\infty} \sum_{j=0}^{n_2} j p(i, j) + (c_1 n_1 - n_2 s_2) \sum_{i=n_1+1}^{\infty} \sum_{j=0}^{n_2} p(i, j) \\ &\quad + (c_1 n_1 + c_2 n_2) \sum_{i=n_1+1}^{\infty} \sum_{j=n_2+1}^{\infty} p(i, j) \end{aligned}$$

or, written as

$$\begin{aligned} M[g_{X,Y}(n_1, n_2)] &= c_1 n_1 \sum_{i=n_1+1}^{\infty} \sum_{j=0}^{\infty} p(i, j) + c_2 n_2 \sum_{i=0}^{\infty} \sum_{j=n_2+1}^{\infty} p(i, j) \\ &\quad - n_1 s_1 \sum_{i=0}^{n_1} \sum_{j=0}^{\infty} p(i, j) - n_2 s_2 \sum_{i=0}^{\infty} \sum_{j=0}^{n_2} p(i, j) \\ &\quad + (c_2 + s_2) \sum_{i=0}^{\infty} \sum_{j=0}^{n_2} j p(i, j) + (c_1 + s_1) \sum_{i=0}^{n_1} \sum_{j=0}^{\infty} i p(i, j) \end{aligned}$$

from which

$$\begin{aligned} M[g_{X,Y}(n_1, n_2)] &= c_1 n_1 \sum_{i=n_1+1}^{\infty} p_X(i) + c_2 n_2 \sum_{j=n_2+1}^{\infty} p_Y(j) \\ &\quad - n_1 s_1 \sum_{i=0}^{n_1} p_X(i) - n_2 s_2 \sum_{j=0}^{n_2} p_Y(j) + (c_2 + s_2) \sum_{j=0}^{n_2} j p_Y(j) + (c_1 + s_1) \sum_{i=0}^{n_1} i p_X(i) \end{aligned}$$

and finally

$$\begin{aligned}
M[g_{X,Y}(n_1, n_2)] &= -n_1 s_1 \sum_{i=0}^{n_1} p_X(i) + c_1 n_1 \sum_{i=n_1+1}^{\infty} p_X(i) + (c_1 + s_1) \sum_{i=0}^{n_1} i p_X(i) \\
&\quad - n_2 s_2 \sum_{j=0}^{n_2} p_Y(j) + c_2 n_2 \sum_{j=n_2+1}^{\infty} p_Y(j) + (c_2 + s_2) \sum_{j=0}^{n_2} j p_Y(j) \\
&= c_1 n_1 \sum_{i=0}^{\infty} p_X(i) + (c_1 + s_1) \sum_{i=0}^{n_1} (i - n_1) p_X(i) \\
&\quad + c_2 n_2 \sum_{j=0}^{\infty} p_Y(j) + (c_2 + s_2) \sum_{j=0}^{n_2} (j - n_2) p_Y(j) \\
&= c_1 n_1 + c_2 n_2 + (c_1 + s_1) \sum_{i=0}^{n_1} (i - n_1) p_X(i) + (c_2 + s_2) \sum_{j=0}^{n_2} (j - n_2) p_Y(j).
\end{aligned}$$

On the other hand, using [3] we can see that

$$M[g_1(n_1, X)] = c_1 n_1 + (c_1 + s_1) \sum_{i=0}^{n_1} (i - n_1) p_X(i)$$

and

$$M[g_2(n_2, Y)] = c_2 n_2 + (c_2 + s_2) \sum_{j=0}^{n_2} (j - n_2) p_Y(j)$$

from which we have $M[g_{X,Y}(n_1, n_2)] = M[g_1(n_1, X)] + M[g_2(n_2, Y)]$, which confirm our intuition.

ii) Setting

$$G(n_1, n_2) := c_1 n_1 + (c_1 + s_1) \sum_{i=0}^{n_1} (i - n_1) p_X(i) + c_2 n_2 + (c_2 + s_2) \sum_{j=0}^{n_2} (j - n_2) p_Y(j)$$

we observe that

$$G(n_1 + 1, n_2 + 1) - G(n_1, n_2) = c_1 - (c_1 + s_1) \sum_{i=0}^{n_1} p_X(i) + c_2 - (c_2 + s_2) \sum_{j=0}^{n_2} p_Y(j).$$

On the other hand, if

$$P(X \leq n_1) = \sum_{i=0}^{n_1} p_X(i) < \frac{c_1}{c_1 + s_1} \text{ and } P(Y \leq n_2) = \sum_{j=0}^{n_2} p_Y(j) < \frac{c_2}{c_2 + s_2} \quad (3.2)$$

it is obvious that

$$G(n_1 + 1, n_2 + 1) - G(n_1, n_2) = M[g_{X,Y}(n_1 + 1, n_2 + 1)] - M[g_{X,Y}(n_1, n_2)] > 0.$$

Absolutely analog

$$G(n_1 + 1, n_2) - G(n_1, n_2) > 0 \text{ and } G(n_1, n_2 + 1) - G(n_1, n_2) > 0.$$

Furthermore, the pair (n_1, n_2) with n_1, n_2 the largest possible that verifies (3.2) achieving the maximum value of $M[g_{X,Y}(n_1, n_2)]$ and thus the profit maximization of the company.

As far as Problem 2 is concerned, to solve (1.1) the analysis splits into two cases:

Case 1: If one of the variables can be written explicitly as a function of the other, then by substituting it in the profit function leads to an unconstrained optimization.

Case 2: If Case 1 does not hold then we can follow the methods presented in Intriligator [1].

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